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SJOGREN'S THEOREM ON DIMENSION SUBGROUPS

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1. Introduction

Let G be a group, $\mathbb{Z}G$ its integral group ring, and $\mathfrak{g}$ its augmentation ideal. The nth dimension subgroup $D_n(G)$ of G (over $\mathbb{Z}$) is defined by $D_n(G)=G\cap(1+\mathfrak{g}^n)$. It is well known and not difficult to prove that $D_n(G)$ contains G_n, the nth term of the lower central series of G. The group $D_n(G)/G_n$ was long known to be periodic; this follows from work of Jennings on dimension subgroups over the rationals (see [5, 11, 14, 15]). It was at one time conjectured that $D_n(G)=G_n$ for all G and n, but this conjecture was refuted in a striking paper of Rips [17], who constructed a finite 2-group G such that $|D_4(G)/G_4|=2$. Tahara [19, 20] has constructed other groups for which certain dimension subgroups differ from the appropriate terms in the lower central series, but to the authors' knowledge, no group G has yet been constructed such that a quotient $D_n(G)/G_n$ contains a non-trival element of odd order.

In a remarkable paper [18], Sjogren has shown that the periodicity of $D_n(G)/G_n$ can be strengthened as follows:

Theorem 1.1 (Sjogren [18]). *Let b_m be the least common multiple of $\{1,2,\dots,m\}$, let $c_1=c_2=1$ and*

$$c_n = \prod_{i=1}^{n-2} b_i^{\binom{n-2}{i}} \quad \textit{for } i\geq 2.$$

Then for any group G, the exponent of $D_n(G)/G_n$ divides c_n.

The exponent obtained here is the same as that given by Sjogren, though this may not be immediately apparent. We have $c_3=b_1=1$, $c_4=b_1^2b_2=2$, $c_5=b_1^3g_2^3b_3=48$, and so on. Thus, $D_4(G)/G_4$ has exponent dividing 2, which is best possible in view of Rips' example; on the other hand, Tahara [20] has shown that the exponent of $D_5(G)/G_5$ always divides 6.

We note that if p is a prime, then p does not divide c_n if $n \leq p+1$, whence we have

Corollary 1.2. *If G is a p-group, then $D_n(G) = G_n$ for $n \leq p+1$.* □

For $n \leq p$, this had been previously obtained by Moran [13].

In this paper, we give a proof of Sjogren's Theorem which we believe is considerably simpler and more accessible than the original. The basic philosophy remains Sjogren's, but there are numerous differences of detail. We have elected not to use Sjogren's spectral sequence terminology, but this is really just a notational difference. We have also dispensed with Sjogren's 'trees', and by considering presentations in which the relators involve only positive powers of the generators, we have avoided using the "Law of Chen, Fox and Lyndon" [2, p. 96, Lemma 3.3], satisfied by the coefficients in the Magnus expansion of an element of a free group.

Previous but less far reaching simplifications of Sjogren's paper have been given by the second author [6,7]. For general background, see [8,14,15]. An interesting application of Sjogren's methods to dimension subgroups of metabelian p-groups has recently been given by Gupta [4].

2. Group rings of free groups

It is well known that $D_n(F) = F_n$ if F is free [3,12,22]. We exploit this as follows. Let F be free, $R \lhd F$, and $G = F/R$. Then $D_n(G) = D_n(F/F_nR)$. For $k = 1, 2, \ldots$, let

$$R(k) = \prod [Y_1, Y_2, \ldots, Y_k],$$

where the product ranges over all choices of $Y_1, \ldots, Y_k$ in which each Y_i is R or F and at least one is R. We obtain the same subgroup by specifying that exactly one Y_i is R, and the three subgroups lemma shows that in fact

$$R(k) = [R, F, \ldots, F]$$

with $k-1$ terms F. We have

$$F_n = F_nR(n) \leq F_nR(n-1) \leq \cdots \leq F_nR(1) = F_nR.$$

We associate to each factor $F/F_nR(k)$ a subgroup related to its nth dimension subgroup, and study how these behave as k decreases.

Corresponding to $R(k)$, we introduce in $\mathbb{Z}F$ the ideal

$$J(k) = \sum \mathfrak{y}_1 \mathfrak{y}_2 \cdots \mathfrak{y}_k \mathbb{Z}F$$

where the sum ranges over all choices of $Y_1, \ldots, Y_k$ in which each is R or F and at least one (or equivalently, exactly one) Y_i is R. Here, augmentation ideals of subgroups are denoted by the small German letter corresponding to the Latin letter denoting the subgroup. Using the identity

$$1-[x,y]=x^{-1}y^{-1}((y-1)(x-1)-(x-1)(y-1))$$

and induction on k, we see that the augmentation ideal of $[Y_1,\dots,Y_k]$ lies in $J(k)$, and hence

$$\mathfrak{r}(k)\le J(k). \tag{2.1}$$

For $k\le n$, let

$$E_{n,k}=(1+\mathfrak{f}^n+J(k))\cap F=(1+\mathfrak{f}^n+J(k))\cap F_k.$$

The second equality holds because $J(k)\le\mathfrak{f}^k$ and $D_k(F)=F_k$. From (2.1) we have $F_nR(k)\le E_{n,k}$, and

$$\begin{aligned} E_{n,k}&=(E_{n,k}\cap F_k)R(k)\ge(E_{n,k}\cap F_{k+1})R(k)\\ &\ge\cdots\ge(E_{n,k}\cap F_{n-1})R(k)\ge F_nR(k). \end{aligned}\tag{2.2}$$

Clearly, $E_{n,1}/F_nR=D_n(F/F_nR)$. Our aim is to obtain bounds on the exponents of the factors in (2.2) by a certain recursive process, and thus eventually to bound the exponent of $D_n(F/F_nR)$.

Let $k\le j<n$ and consider the factor $(E_{n,k}\cap F_j)R(k)/(E_{n,k}\cap F_{j+1})R(k)$. We have $E_{n,k}=(1+\mathfrak{f}^n+J(k))\cap F\le(1+\mathfrak{f}^{j+1}+J(k))\cap F=E_{j+1,k}$. Also

$$((E_{n,k}\cap F_j)R(k))\cap F_{j+1}R(k)\le E_{n,k}\cap F_{j+1}R(k)=(E_{n,k}\cap F_{j+1})R(k)$$

by the modular law. The reverse equality is trivially true. We now have

Lemma 2.1. *Inclusion induces an embedding of*

$$(E_{n,k}\cap F_j)R(k)/(E_{n,k}\cap F_{j+1})R(k)$$

into

$$(E_{j+1,k}\cap F_j)R(k)/F_{j+1}R(k),\quad k\le j<n.\qquad\square$$

Fixing k, we have a series like (2.2) for each $j<n$, obtained by replacing n by $j+1$. Lemma 2.2 tells us that if we can bound the exponent of the last factor in each of these, then we have a bound for the exponent of $E_{n,k}/F_nR(k)$. Let $a_{j+1,k}$ denote the exponent of $(E_{j+1,k}\cap F_j)R(k)/F_{j+1}R(k)$ $(k\le j)$. We have

Lemma 2.2. *The exponent of $E_{n,k}/F_nR(k)$ divides $\prod_{j=k+1}^n a_{j,k}$.* $\square$

The characterization of dimension subgroup of free groups gives $a_{n,n}=1$. We shall prove

Lemma 2.3. $a_{n,n-1}=1$.

Thus the first inclusion in (2.2) is actually an equality.

For fixed n, the behaviour of (2.2) also improves as k increases, since $F/F_nR(k)$

comes closer to a free group. This will be exploited in the main step in the proof of Sjogren's Theorem, which is

Lemma 2.4. *Let F be freely generated by $\{x_i : i \in I\}$ and R be the normal closure in F of $\{r_l : l \in L\}$. Suppose that each r_l is a product of non-negative powers of the x_i. If $k<n$ and $x \in E_{n,k} \cap F_{n-1}$, then there exists $r \in R(k)$ such that $x^{b_k} r \in E_{n,k+1}$.*

This lemma is actually true without the assumption on the form of the relators r_l. However, this assumption makes the proof considerably simpler, as it avoids the Law of Chen, Fox and Lyndon, which plays a central role in Sjogren's original paper. Since this version of the lemma is all we need for Sjogren's Theorem, we confine ourselves to it here. For a proof of the general case, see [9,21]. That in [7] is incorrect.

Continuing with the proof of Sjogren's Theorem, we note that every group has a presentation in which each relator is a product of non-negative powers of the generators. Such a presentation can be obtained from an arbitrary presentation by simply adjoining extra generators corresponding to the inverses of the original ones. We now assume that our presentation satisfies this condition.

Let $y = x^{b_k} r$, where x is as in Lemma 2.4. By Lemmas 2.2 and 2.3, we have $y^s \in F_n R(k+1) \le F_n R(k)$, where $s = \prod_{j=k+3}^{n} a_{j,k+1}$ (empty products, as usual, are interpreted as 1). Hence $x^{b_k s} \in F_n R(k)$. Thus,

$$\textit{if } k \le n-2, \textit{ then } a_{n,k} \textit{ divides } b_k \prod_{j=k+3}^{n} a_{j,k+1}. \tag{2.3}$$

From this, we can bound $a_{n,k}$ by induction on $n-k$. Evidently it divides a number of the form $\prod_{l=k}^{n-2} b_l^{\alpha_l(n,k)}$, where

$$\begin{aligned} \alpha_k(n,k) &= 1 \quad \text{if } n \ge k+2, \\ \alpha_k(n,l) &= \sum_{j=l+3}^{n} \alpha_k(j,l+1), \qquad l<k\le n-2, \\ &= \alpha_k(n-1,l) + \alpha_k(n,l+1), \quad l<k\le n-3. \end{aligned} \tag{2.4}$$

From these equations we see by induction that

$$\alpha_k(n,l) = \binom{n-l-2}{n-k-2}, \quad 1 \le l \le k \le n-2.$$

From Lemma 2.2, the exponent of $D_n(F/F_nR) = E_{n,1}/F_nR$ divides $\prod_{j=2}^{n} a_{j,1}$. This divides a product in which b_k occurs with exponent $\sum_{j=3}^{n} \alpha_k(j,1)$ (note $a_{2,1}=1$), which we see, by putting $l=0$ in (2.4), is equal to

$$\binom{n-2}{n-k-2} = \binom{n-2}{k}.$$

Thus, the exponent of $D_n(F/F_nR)$ divides $\prod_{k=1}^{n-2} b_k^{\binom{n-2}{k}}$, giving Sjogren's Theorem.

The proof of Lemma 2.3 is quite straightforward and we give it immediately. Lemma 2.4 requires some Lie-theoretic machinery and we deal with it later.

Proof of Lemma 2.3. We have an element $x \in F_{n-1} \cap (1 + J(n-1) + \mathfrak{f}^n)$ and we have to prove that $x \in F_n R(n-1)$. Let $\{x_i : i \in I\}$ be a set of free generators of F. Then, modulo $\mathfrak{f}^n$, every element of $\mathbb{Z}F$ can be uniquely represented as a $\mathbb{Z}$-linear combination of monomials of degree $\le n-1$ in the elements $\tilde{x}_i = x_i - 1$, treated as non-commuting variables. We have the map

$$\phi : y \to \tilde{y} = y - 1, \quad y \in F$$

of F into $\mathbb{Z}F$, and if $y_1, \dots, y_t \in F$, then

$$\phi([y_1, \dots, y_t]) \equiv (\tilde{y}_1, \dots, \tilde{y}_t) \bmod \mathfrak{f}^{t+1}, \tag{2.5}$$

where $(u, v) = uv - vu$ and the commutators are left-normed. Thus, since $x \in F_{n-1}$, $\tilde{x}$ is a Lie element of $\mathfrak{f}^{n-1}/\mathfrak{f}^n$, that is, an element of the Lie subring of $\mathbb{Z}F/\mathfrak{f}^n$ generated by the $\tilde{x}_i$.

Let $V = \bigoplus_{i \in I} \mathbb{Z}\tilde{x}_i$ and $W = (\mathfrak{f}^2 + \mathfrak{r}\mathbb{Z}F) \cap V$. Then, modulo $\mathfrak{f}^n$, every element of $J(n-1)$ is congruent to a linear combination of monomials of degree $n-1$ in which $n-2$ factors have the form $\tilde{x}_i$ and the other comes from W, and this is true in particular for $x-1$. We can choose a finite subset J of I such that all the $\tilde{x}_i$ and elements of W appearing, and also the $\tilde{x}_i$ appearing in an expression for $\tilde{x}$ as a Lie element of $\mathfrak{f}^{n-1}/\mathfrak{f}^n$, lie in $T = \bigoplus_{j \in J} \mathbb{Z}\tilde{x}_j$. We can then choose a basis $y_1, \dots, y_t$ of T such that $W \cap T$ has basis $z_1 = n_1 y_1, \dots, z_s = n_s y_s$, where $s \le t$ and $n_1, \dots, n_s$ are non-zero integers satisfying $n_1 \mid n_2 \mid \cdots \mid n_s$. Now we construct a sequence of basic commutators in $y_1, \dots, y_t$. Then, modulo $\mathfrak{f}^n$, $x-1$ is congruent to a $\mathbb{Z}$-linear combination of basic commutators of weight $n-1$ in $y_1, \dots, y_t$, and we can write uniquely

$$x - 1 \equiv \xi_1 + \cdots + \xi_{t-1} \bmod \mathfrak{f}^n, \tag{2.6}$$

where ξ_j is a linear combination of basic commutators of weight $n-1$ involving y_j but not involving $y_1, \dots, y_{j-1}$.

On the other hand, $x-1$ is congruent, modulo $\mathfrak{f}^n$, to a linear combination of associative monomials of degree $n-1$ in the y_i and z_i, each involving exactly one z_i, and so

$$x - 1 \equiv \eta_1 + \cdots + \eta_s \bmod \mathfrak{f}^n \tag{2.7}$$

where η_j is a linear combination of monomials involving y_j or z_j, exactly one z_i, but no y_k or z_k for which $k<j$. Replacing each z_j by $n_j y_j$ and using the divisibility properties of the n_l, we see that $\eta_j = n_j \eta_j'$, where η_j' is a linear combination of monomials in the y_l involving y_j but not $y_1, \dots, y_{j-1}$. Comparing (2.6) and (2.7) shows that $\xi_j = n_j \eta_j'$ $(1 \le j \le s)$ and $\xi_{s+1} = \cdots = \xi_{t-1} = 0$. Since the basic commutators of weight $n-1$ form part of a $\mathbb{Z}$-basis of $\mathfrak{f}^{n-1}/\mathfrak{f}^n$, it follows that $\xi_j = n_j \xi_j'$, where ξ_j' is a linear combination of basic commutators in the y_l involving y_j but not $y_1, \dots, y_{j-1}$. Thus, ξ_j is a linear combination of Lie products of weight $n-1$ in z_j,

$y_{j+1},\dots,y_t$, each involving exactly one z_j. There exist elements $u_1,\dots,u_t \in F$ and $v_1,\dots,v_s \in R$ such that $y_j \equiv \tilde{u}_j$ and $z_j \equiv \tilde{v}_j \bmod \mathfrak{f}^2$. Then any Lie product of weight $n-1$ in the y_j and z_j is unchanged, modulo $\mathfrak{f}^n$, by replacing y_j and z_j by $\tilde{u}_j$ and $\tilde{v}_j$ respectively. Thus $x-1$ is congruent modulo $\mathfrak{f}^n$ to a linear combination of Lie products of weight $n-1$ in the $\tilde{u}_j$ and $\tilde{v}_j$, each involving exactly one $\tilde{v}_j$. By (2.5), $x-1 \equiv w-1 \bmod \mathfrak{f}^n$, where $w \in R(n-1)$. Hence $xw^{-1} \in F_n$, by the characterization of dimension subgroups of free groups, and $x \in F_n R(n-1)$. □

3. The Sjogren maps ψ and $\hat{\psi}$

Certain maps introduced by Sjogren play a crucial role in the proof of Lemma 2.4.

Let A be the free associative algebra over $\mathbb{Q}$ generated freely by a countable number of elements $y_1, y_2, \dots$. For $v \geq 1$ and $t \geq 1$, we obtain an element $\sigma_{v,t}(y_1,\dots,y_t) \in A$ by taking from the element

$$((-1)^{v+1}/v)((y_1+1)\cdots(y_t+1)-1)^v \tag{3.1}$$

of A, all the monomials that involve all of $y_1,\dots,y_t$, together with their coefficients. We put $\sigma_{v,0}=0$. Thus, for example,

$$\sigma_{2,2}(y_1,y_2) = -\tfrac{1}{2}(y_1y_2 + y_1^2y_2 + y_2y_1 + y_2y_1y_2 + y_1y_2y_1 + y_1y_2^2 + y_1y_2y_1y_2).$$

The expression (3.1) is of course taken from the formal expansion

$$\log((y_1+1)\cdots(y_t+1)) = \sum_{v=1}^{\infty} ((-1)^{v+1}/v)((y_1+1)\cdots(y_t+1)-1)^v. \tag{3.2}$$

We also define

$$\hat{\sigma}_{v.t}(y_1,\dots,y_t)$$

to be the element obtained by taking the terms of degree exactly t from $\sigma_{v,t}(x_1,\dots,x_t)$. Thus

$$\hat{\sigma}_{2,2}(y_1,y_2) = -\tfrac{1}{2}(y_1y_2 + y_2y_1). \tag{3.3}$$

Now given an arbitrary free associative algebra B over $\mathbb{Q}$ (or any field of characteristic zero), together with a set of free generators $x_1, x_2, \dots$ and an integer $l \geq 1$, we define $\mathbb{Q}$-linear maps ψ_v, $\hat{\psi}_v$ $(v \geq 1)$, ψ and $\hat{\psi}$ from B to B by specifying their effect on monomials as follows:

$$\psi_v(x_{i_1}\cdots x_{i_t}) = \sigma_{v,t}(x_{i_1},\dots,x_{i_t}), \quad t \geq 0; \tag{3.4}$$

$$\hat{\psi}_v(x_{i_1}\cdots x_{i_t}) = \hat{\sigma}_{v,t}(x_{i_1},\dots,x_{i_t}), \quad t \geq 0; \tag{3.5}$$

$$\psi = \psi^{(l)} \equiv \sum_{v=1}^{l} \psi_v; \tag{3.6}$$

$$\hat{\psi} = \hat{\psi}^{(l)} = \sum_{\nu=1}^{l} \hat{\psi}_\nu. \tag{3.7}$$

Clearly

$$\psi(x_{i_1} \cdots x_{i_t}) = \hat{\psi}(x_{i_1} \cdots x_{i_t}) + \text{terms of degree} \geq t+1. \tag{3.8}$$

Also, recalling that b_l is the least common multiple of $\{1,2,\ldots,l\}$, we have

$$b_l\psi \textit{ and } b_l\hat{\psi} \textit{ map } \mathbb{Z}[x_1,x_2,\ldots] \textit{ into itself.} \tag{3.9}$$

We can think of ψ_ν for example as being obtained by substituting $x_{i_1},\ldots,x_{i_t}$ for $y_1,\ldots,y_t$ in (3.2) and then taking from the νth term the monomials that 'formally' involve all of $x_{i_1},\ldots,x_{i_t}$. We note that $\hat{\psi}_1(y_1y_2)=y_1y_2$, so from (3.3),

$$\hat{\psi}^{(2)}(y_1y_2) = \tfrac{1}{2}(y_1y_2-y_2y_1) = \tfrac{1}{2}(y_1, y_2). \tag{3.10}$$

A fundamental property of the maps ψ_ν is the following. If $(i_1,i_2,\ldots,i_s)$ is a finite sequence of integers, it will be convenient to write

$$x_I = x_{i_1}x_{i_2}\cdots x_{i_s}.$$

Lemma 3.1. *Let B be a free associative algebra over $\mathbb{Q}$ freely generated by $x_1,x_2,\ldots$, let $(j_1,j_2,\ldots,j_r)$ be a finite subsequence of $(1,2,3,\ldots)$ $(r>0)$ and let $\delta : B\to B$ be the $\mathbb{Q}$-algebra homomorphism given by $\delta(x_i)=x_i$ $(i\neq j)$, $\delta(x_j)=(x_{j_1}+1)\cdots(x_{j_r}+1)-1$, where $j\geq 1$ is a fixed integer. Let ξ be any monomial in the x_i's which involves x_j exactly once. Then*

$$\delta\psi_\nu(\xi) = \psi_\nu\delta(\xi). \tag{3.11}$$

Remark. This is a special case of a more general result, but since the above is all we need for Sjogren's Theorem, and since our aim is to give as simple a proof of that theorem as possible, we confine ourselves to it. It is, for example, not necessary to assume that ξ involves x_j exactly once. The element $\delta(x_j)$ above should be viewed as the Magnus expansion of a product of positive powers of the generators of a free group, up to terms of degree r. The corresponding result for words involving negative exponents is also true, and was essentially observed by Sjogren. A statement and proof of the more general result can be found in [9], but for a much more illuminating account in a more general context, see [21].

Proof of Lemma 3.1. Let

$$\xi = x_{i_1}\cdots x_{i_l}x_jx_{i_{l+1}}\cdots x_{i_m}, \tag{3.12}$$

where $l,m\geq 0$ and $j\notin\{i_1,\ldots,i_m\}$. Let B_1 be the free associative $\mathbb{Q}$-algebra freely generated by $y_1,\ldots,y_m$, $z_1,\ldots,z_r$ and s, and let δ' be the $\mathbb{Q}$-algebra homomorphism from B_1 to itself which fixes all the generators except s and sends the latter to $(z_1+1)\cdots(z_r+1)-1$. Then with ξ as in (3.12), (3.11) follows from

$$\psi_\nu \delta'(y_1 \cdots y_l s y_{l+1} \cdots y_m) = \delta' \psi_\nu (y_1 \cdots y_l s y_{l+1} \cdots y_m). \tag{3.13}$$

To see this, consider the $\mathbb{Q}$-algebra homomorphism (or 'specialization') $\varepsilon : B_1 \to B$ given by $\varepsilon(y_t) = x_{i_t}$ $(1 \le t \le m)$, $\varepsilon(z_t) = x_{j_t}$ $(1 \le t \le r)$ and $\varepsilon(s) = x_j$. We clearly have $\psi_\nu \varepsilon = \varepsilon \psi_\nu$, where the two versions of ψ_ν, on B and B_1 respectively, are calculated with respect to the given systems of generators, and $\delta\varepsilon = \varepsilon\delta'$ on the subalgebra of B_1 generated by $y_1, \dots, y_m$ and s. Writing $\eta = y_1 \cdots y_l s y_{l+1} \cdots y_m$, we have

$$\begin{aligned} \psi_\nu \delta(\xi) &= \psi_\nu \delta \varepsilon(\eta) = \psi_\nu \varepsilon \delta'(\eta) = \varepsilon \psi_\nu \delta'(\eta) \\ &= \varepsilon \delta' \psi_\nu(\eta) \text{ (from (3.13)} = \delta \psi_\nu \varepsilon(\eta) = \delta \psi_\nu(\xi). \end{aligned}$$

Let $\phi_\nu = (-1)^{\nu+1} \nu \psi_\nu$. Evidently, in proving (3.13), we may replace ψ_ν by ϕ_ν. For brevity, write $I = (1, 2, \dots, l)$ and $I' = (l+1, \dots, m)$. Then $\phi_\nu \delta'(y_I s y_{I'}) = \phi_\nu(y_I((z_1+1) \cdots (z_r+1) - 1) y_{I'})$, and since ϕ_ν is linear, we have

$$\phi_\nu \delta'(y_I s y_{I'}) = \sum_K \phi_\nu(y_I z_K y_{I'}), \tag{3.14}$$

where the sum is over all non-empty subsequences K of $(1, 2, \dots, r)$.

On the other hand, to form $\delta' \phi_\nu(y_I s y_{I'})$, we first write down

$$((y_1+1) \cdots (y_l+1)(s+1)(y_{l+1}+1) \cdots (y_m+1) - 1)^\nu$$

and take the terms that involve all the generators. We then substitute $(z_1+1) \cdots (z_r+1) - 1$ for s. This amounts to taking all the terms from

$$((y_1+1) \cdots (y_l+1)(z_1+1) \cdots (z_r+1)(y_{l+1}+1) \cdots (y_m+1) - 1)^\nu$$

that involve all the y's and at least one z. For each non-empty subsequence K of $(1, 2, \dots, r)$, let η_K denote the sum of the terms so obtained that involve exactly those z_k for which $k \in K$. Then

$$\eta_K = \phi_\nu(y_I z_K y_{I'})$$

and so

$$\delta' \phi_\nu(y_I s y_{I'}) = \sum_K \eta_K = \sum_K \phi_\nu(y_I z_K y_{I'}),$$

where the sum is over all non-empty subsequences K of $(1, 2, \dots, r)$. Now (3.13) follows from this and (3.14). □

Remark. The maps $\hat{\psi}_\nu$ do not possess the property (3.11). For example, taking $\delta(s) = (x+1)(y+1) - 1$, we have, using (3.10), that $\hat{\psi}_2 \delta(s) = \hat{\psi}_2(x + y + xy) = -\frac{1}{2}(xy + yx)$ while $\delta \hat{\psi}_2(s) = 0$. In this connection, however, see [21].

The second fundamental property of $\hat{\psi}$ concerns its behaviour with respect to Lie elements.

Lemma 3.2. *Let A be the free associative $\mathbb{Q}$-algebra freely generated by $y_1, y_2, \dots$.*

(i) *If a is a homogeneous element of degree $t \le l$ in A, then $\hat{\psi}(a)$ is a Lie element of A.*

(ii) *If a is any Lie element of A, then $\hat{\psi}(a) = a$.*

Proof. (i) It suffices to assume that $a = y_1 y_2 \cdots y_t$, since the result for a general monomial follows by specialization, and that for a general element by linearity. In that case, $\hat{\psi}(a)$ is equal to that part of the expansion of $y = \log(e^{y_1} \cdots e^{y_t})$ which is a linear combination of monomials of degree t, each involving each y_i exactly once. By the Baker–Campbell–Hausdorff formula [10], y is a Lie element, and hence so is $\hat{\psi}(a)$.

(ii) Using a specialization argument again, we may assume that $a = (a_1, a_2, \dots, a_t)$, where $t \ge 2$, the elements $a_1, a_2, \dots, a_t$ are distinct free generators of a free associative $\mathbb{Q}$-algebra and the commutator is left-normed. When $t = 2$, we have $\hat{\psi}_1(a) = a$ since $\hat{\psi}_1$ is the identity map. From (3.3), $\hat{\psi}_2(a_1 a_2 - a_2 a_1) = 0$, and $\hat{\psi}_3(a) = \cdots = \hat{\psi}_l(a) = 0$ since a has degree 2. Now let $t > 2$ and use induction on t. Take an additional free generator b, and let $\delta = \delta_1 - \delta_2$, where δ_1, δ_2 are the $\mathbb{Q}$-algebra homomorphisms of $\mathbb{Q}[b, a_1, a_2, \dots]$ into itself that fix the a_i and send b respectively to $(a_1+1)(a_2+1)$ and $(a_2+1)(a_1+1)$. Then $\delta(b) = (a_1, a_2)$, and the effect of δ on an associative monomial involving exactly one b is to replace that b by (a_1, a_2). Thus $\delta((b, a_3, \dots, a_t)) = a$. By Lemma 3.1,

$$\psi\delta((b, a_3, \dots, a_t)) = \delta\psi((b, a_3, \dots, a_t)). \tag{3.15}$$

Now $\hat{\psi}\delta((b, a_3, \dots, a_t))$ is obtained by taking terms of degree t from the left-hand side of this. Since any monomial that contains b exactly once has its degree raised by exactly one on applying δ, this corresponds to taking the terms of degree $t-1$ from $\psi((b, a_3, \dots, a_t))$, that is, replacing ψ by $\hat{\psi}$ on the right-hand side of (3.15). Thus

$$\begin{aligned} \hat{\psi}(a) &= \hat{\psi}\delta((b, a_3, \dots, a_t)) = \delta\hat{\psi}((b, a_3, \dots, a_t)) \\ &= \delta((b, a_3, \dots, a_t)) \text{ (using induction)} = a. \quad \square \end{aligned}$$

Remark. Sjogren's original proof of Lemma 3.2(ii) used a characterization of Lie elements due to Ree [16]. The above argument uses an idea due to Gupta [4], and we are grateful for his permission to use it.

Proof of Lemma 2.4. We have $G = F/R$, where F is freely generated by $\{x_i : i \in I\}$ and R is the normal closure of $\{r_l : l \in L\}$. Each r_l is the product of non-negative powers of the x_i. We also have an element $x \in E_{n,k} \cap F_{n-1}$, where $k < n$, and have to show that there exists $r \in R(k)$ such that $x^{b_k} r \in E_{n,k+1}$.

Now $\mathbb{Z}F/\mathfrak{f}^n$ can be viewed as the free associative ring generated by the elements $\tilde{x}_i = (x_i - 1) + \mathfrak{f}^n$ $(i \in I)$, in which monomials of degree $>n$ are equated to zero. Writing $\tilde{r}_l$ for the coset of $r_l - 1$ $(l \in L)$, we have that the monomials of degree at least k in the $\tilde{x}_i$ and $\tilde{r}_l$, involving exactly one $\tilde{r}_l$, additively span a two-sided ideal of $\mathbb{Z}F/\mathfrak{f}^n$. This ideal is $J(k)+\mathfrak{f}^n/\mathfrak{f}^n$.

If $r_l = x_{j_1} \cdots x_{j_r}$, then we have

$$\tilde{r}_l = (\tilde{x}_{j_1}+1)\cdots(\tilde{x}_{j_r}+1)-1. \tag{3.16}$$

Now let S be a free group freely generated by elements s_l $(l \in L)$, and let $F^* = F * S$. We think of $\mathbb{Z}F^*/\mathfrak{f}^{*n}$ as the free associative ring generated by elements $\tilde{x}_i = x_i - 1 + \mathfrak{f}^{*n}$ and $\tilde{s}_l = (s_l - 1) + \mathfrak{f}^{*n}$, with monomials of degree $> n$ equated to zero. Let $\beta : F^* \to F$ be the homomorphism given by $\beta(x_i) = x_i$ $(i \in I)$ and $\beta(s_l) = r_l$ $(l \in L)$. This induces the following commutative square,

$$\begin{tikzcd}
F^* \arrow[r, "\beta"] \arrow[d] & F \arrow[d] \\
\mathbb{Z}F^*/\mathfrak{f}^{*n} \arrow[r, "\delta"'] & \mathbb{Z}F/\mathfrak{f}^n
\end{tikzcd}$$

in which the vertical maps are the natural homomorphisms, $\delta(\tilde{x}_i) = \tilde{x}_i$ $(i \in I)$, and from (3.16), $\delta(\tilde{s}_l)$ has the form

$$\delta(\tilde{s}_l) = (\tilde{x}_{j_1}+1)\cdots(\tilde{x}_{j_r}+1)-1, \quad l \in L. \tag{3.17}$$

Here the values of r and $j_1, \ldots, j_r$ will of course depend on l.

Let $B^* = (\mathbb{Z}F^*/\mathfrak{f}^{*n}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $B = (\mathbb{Z}F/\mathfrak{f}^n) \otimes_{\mathbb{Z}} \mathbb{Q}$. The map δ extends to a homomorphism from B^* to B. On B^* and B we have the maps $\psi = \sum_{\nu=1}^{k} \psi_\nu$ and $\hat{\psi} = \sum_{\nu=1}^{k} \hat{\psi}_\nu$, defined with respect to the given sets of generators. Let W be the $\mathbb{Q}$-subspace of B^* spanned by the monomials involving exactly one $\tilde{s}_l$. Then it follows from Lemma 3.1 that

$$\psi\delta(\xi) = \delta\psi(\xi) \quad \text{if } \xi \in W. \tag{3.18}$$

Now since $x \in F_{n-1}$, we have that $(x-1)+\mathfrak{f}^n$ is a Lie element of $\mathfrak{f}^{n-1}/\mathfrak{f}^n$. Also since $x-1 \in J(k)+\mathfrak{f}^n$, $x-1+\mathfrak{f}^n$ is a $\mathbb{Z}$-linear combination of monomials in the $\tilde{x}_i$ and $\tilde{r}_l$, of degree at least k, and each involving exactly one $\tilde{r}_l$. Thus, $x-1+\mathfrak{f}^n = \delta(a)$, where a is simply obtained by replacing each $\tilde{r}_l$ with $\tilde{s}_l$. From (3.9), $\psi(b_k a) \in \mathbb{Z}F/\mathfrak{f}^n$, and from (3.8),

$$\psi(b_k a) = \hat{\psi}(b_k a)_k + w, \tag{3.19}$$

where w is a $\mathbb{Z}$-linear combination of monomials in the $\tilde{x}_i$ and $\tilde{s}_l$, of degree at least $k+1$, each involving at least one $\tilde{s}_l$, and $\hat{\psi}(b_k a)_k$ is the homogeneous component of degree k of $\hat{\psi}(b_k a)$. By Lemma 3.2, $\hat{\psi}(b_k a)_k$ is a Lie element of $\mathbb{Z}F^*/\mathfrak{f}^{*n}$, and so by (2.5) we have

$$\hat{\psi}(b_k a)_k = (z-1)+u, \tag{3.20}$$

where z is the coset of a product of group commutators of weight k in the x_i and

s_l, each involving exactly one s_l, and u is a $\mathbb{Z}$-linear combination of monomials of degree at least $k+1$, each involving some $\tilde{s}_l$. Thus, from (3.19) and (3.20),

$$\psi(b_k a) = (z-1) + v \tag{3.21}$$

where $v \in W$ and $\delta(v) \in J(k+1) + \mathfrak{f}^n/\mathfrak{f}^n$. We apply the map δ to (3.21). Since $b_k a \in W$, (3.18) shows that the left-hand side becomes

$$\delta\psi(b_k a) = \psi\delta(b_k a) = \psi(b_k(x-1) + \mathfrak{f}^n).$$

Now ψ and $\hat{\psi}$ agree on $\mathfrak{f}^{n-1}/\mathfrak{f}^n$ and $\hat{\psi}(b_k(x-1)+\mathfrak{f}^n) = b_k(x-1)+\mathfrak{f}^n$, by Lemma 3.2(ii). Thus, applying δ to (3.21) gives

$$b_k(x-1) + \mathfrak{f}^n = (r^* - 1) + \delta(v),$$

where r^* is obtained by replacing each s_l by r_l in z, and so belongs to $R(k)$, and $\delta(v) \in J(k+1) + \mathfrak{f}^n/\mathfrak{f}^n$. This gives $x^{b_k}r - 1 \in J(k+1) + \mathfrak{f}^n$, completing the proof of Lemma 3.1 and Sjogren's Theorem. □

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